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Triple I Method of Fuzzy Reasoning

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Abstract—The theory of the triple I method with total inference rules of fuzzy reasoning is investigated by using Zadeh's implication operator R_z. The computational formulae for both fuzzy modus ponens (FMP) and fuzzy modus tollens (FMT) are obtained. The reversibility properties for FMP and FMT are analyzed and the reversibility criteria are given. We also investigated the generalized problem of the triple I method and obtained the formulae for the α-triple I FMP and the α-triple I FMT. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Fuzzy reasoning, Triple I method, Zadeh's implication operator.

1. INTRODUCTION

Since Zadeh introduced the compositional rule of fuzzy inference (CRI) [1–3], various studies and successful applications in fuzzy logic have been carried out [4–9]. Fuzzy reasoning becomes the theoretical basis and an important tool for designing fuzzy controllers. For example, for multi-input-single-output (MISO) linear systems, the mathematical model of fuzzy reasoning can be represented by

$$\begin{array}{llcl}
\text{rule base} & R_1 : A_1 & \text{and} & B_1 \to C_1 \\
 & R_2 : A_2 & \text{and} & B_2 \to C_2 \\
 & & \cdots & \\
 & R_n : A_n & \text{and} & B_n \to C_n \\
\text{for given} & A^* & \text{and} & B^* \\
\hline
\text{to determine} & & & C^*.
\end{array} \tag{1}$$

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Clearly, it is based on fuzzy modus ponens, which can be represented as

$$\begin{array}{ll} \text{rule} & A \to B \\ \text{for given} & A^* \\ \hline \text{to determine} & B^*, \end{array} \tag{2}$$

where A and A^* are the fuzzy sets in domain X, and B and B^* are the fuzzy sets in domain Y.

According to Zadeh's method of CRI, the fuzzy set B^* in (2) may be determined by

$$B^*(y) = A^*(x) \circ R_z(A(x), B(y)), \qquad y \in Y, \tag{3}$$

where the variable $R_z : [0,1]^2 \to [0,1]$ is Zadeh's implication operator defined as

$$R_z(a,b) = \begin{cases} a' \vee a, & a \le b, \\ a' \vee b, & a > b, \end{cases} \tag{4}$$

where $a' = 1 - a$, $a \in [0,1]$. Notice that there are two kinds of fuzzy logic operators in (3), the compound operator and the implication operator "R_z". These operators have been defined by various different methods [10].

The opposite form of fuzzy modus ponens is fuzzy modus tollens, which can be expressed as

$$\begin{array}{ll} \text{rule} & A \to B \\ \text{for given} & B^* \\ \hline \text{to determine} & A^*, \end{array} \tag{5}$$

where A and A^* are fuzzy sets in domain X, and B and B^* are the fuzzy sets in the domain Y. Again, A^* in (5) can be determined according to Zadeh's method of CRI.

CRI has been widely applied successfully in various fields of industrial control. However, from the standpoint of logic semantics, there exist several problems in applying the method of CRI. For example, the approach does not possess the reversibility property [11]. Furthermore, Li [12] has proved that, mathematically, the ordinary fuzzy control method depending on CRI may be regarded as a certain interpolation method. In fact, CRI is such an implication method that adopts fuzzy reasoning only once, and other implications are simply replaced by the compound method. To improve the method of CRI, the triple I method with total inference rules of fuzzy reasoning was proposed by Wang [11], where the implication operator $R_0 : [0,1]^2 \to [0,1]$,

$$R_0(a,b) = \begin{cases} 1, & a \le b, \\ a' \vee b, & a > b, \end{cases}$$

was adopted to replace Zadeh's implication operator R_z. The computational formulae for the triple I FMP and FMT were also given. Furthermore, the triple I method was generated to the generalized form of the α-triple I method, and the formulae for FMP and FMT for this generalized α-triple I method were obtained.

The basic idea of the triple I method can be summarized as follows. For $A, B \in \mathcal{F}(X)$ and $A^*(B^*) \in \mathcal{F}(Y)$, our purpose is to seek an optimal $B^*(A^*) \in \mathcal{F}(Y)$ such that $A \to B$ completely sustains $A^* \to B^*$; that is,

$$M(x,y) = (A(x) \to B(y)) \to (A^*(x) \to B^*(y)) \tag{6}$$

has the maximal possible value whenever $x \in X$ and $y \in Y$, where $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ denote, respectively, the collections of all fuzzy subsets of X and Y.

The generalized form of the triple I method is regarded as the following optimization problem. For any $\alpha \in [0,1]$ with known A, B and $A^*(B^*)$, we wish to seek the optimal $B^*(A^*)$ such that

$$(A(x) \to B(y)) \to (A^*(x) \to B^*(y)) \ge \alpha, \tag{7}$$

whenever $x \in X$ and $y \in Y$.

Since Zadeh's implication operator R_z has been widely adopted, the triple I method of fuzzy reasoning is investigated in this paper by using this implication operator. The formulae for FMP and FMT are obtained, and the reversibility properties are analyzed. The reversibility criteria are also given. In addition, the general formulae for the generalized α-triple I FMP and FMT are obtained. The results of this paper offer powerful approaches for realizing the total reference rules in performance index and in fuzzy control.

2. TRIPLE I METHOD FOR ZADEH'S IMPLICATION OPERATOR R_z

We will use Zadeh's implication operator R_z defined by (4). However, we must first summarize the basic principles and theories of the triple I method given by Wang [11].

Principle of Triple I FMP

Suppose that X and Y are nonempty sets, $A, A^* \in \mathcal{F}(X)$ and $B \in \mathcal{F}(Y)$. Then $B^* \in \mathcal{F}(Y)$ satisfying (2) is the minimal fuzzy set that makes $M(x,y)$ in (6) to take its maximum. For Zadeh's implication operator R_z, B^* is the minimal fuzzy set that makes $M(x,y)$ in (6) take its maximum $R_z(A(x), B(y)) \to A^*(x) \vee (A^*(x))'$.

REMARK 1. By the definition of Zadeh's operator R_z, it is clear that if we take $B^*(y) \equiv 1$, then $M(x,y)$ in (6) always takes its maximum $R_z(A(x), B(y)) \to A^*(x) \vee (A^*(x))'$. We will seek the minimal fuzzy set in $\mathcal{F}(Y)$ which makes $M(x,y)$ in (6) take its maximum $R_z(A(x), B(y)) \to A^*(x) \vee (A^*(x))'$.

According to the above-mentioned principle, we have the following triple I FMP method.

THEOREM 1. TRIPLE I FMP FOR R_z. *Suppose that X, Y are nonempty sets, $A, A^* \in \mathcal{F}(X)$ and $B \in \mathcal{F}(Y)$. Then for Zadeh's operator R_z, the minimal fuzzy set $B^* \in \mathcal{F}(y)$ that makes $M(x,y)$ in (6) take its maximum is determined by*

$$B^*(y) = \sup_{\substack{x \in E_y \\ R_z(A(x),B(y))>1/2}} [A^*(x) \wedge R_z(A(x), B(y))], \qquad y \in Y, \tag{8}$$

where

$$E_y = \{x \in X \mid (A^*(x))' < R_z(A(x), B(y))\}. \tag{9}$$

PROOF. See Theorem 2 in [11]. ∎

REMARK 2. It is easy to see that for any $B_0^*(y) \in \mathcal{F}(Y)$ satisfying $B^*(y) \leq B_0^*(y) \leq 1$, then $B_0^*(y)$ will make $M(x,y)$ in (6) always take its maximum $R_z(A(x), B(y)) \to A^*(x) \vee (A^*(x))'$, where $B^*(y)$ is determined by (8).

In addition, if we use the method of CRI for FMP with Zadeh's implication operator, then from (3) we can determine $B^* \in \mathcal{F}(Y)$ as follows:

$$B^*(y) = \sup_{x \in X} [A^*(x) \wedge R_z(A(x), B(y))], \qquad y \in Y.$$

According to the principle of the triple I FMP, it can be seen that by using Zadeh's implication operator, the result obtained by the triple I FMP method in Theorem 1 is more exact than that by the method of CRI.

Similarly, for the problem of triple I FMT, the following principle can be obtained.

Principle of Triple I FMT

Suppose that X and Y are nonempty sets, $A \in \mathcal{F}(X), B, B^* \in \mathcal{F}(Y)$. Then $A^* \in \mathcal{F}(X)$ satisfying (5) is the maximal fuzzy set that makes $M(x,y)$ in (6) take its maximum. For Zadeh's implication operator R_z, A^* is the maximal fuzzy set that makes $M(x,y)$ in (6) take its maximum $R_z(A(x),B(y)) \vee R'_z(A(x),B(y))$.

REMARK 3. By the definition of Zadeh's operator R_z, it is clear that by taking $A^*(x) \equiv 0$, then $M(x,y)$ in (6) always takes its maximum $R_z(A(x),B(y)) \vee R'_z(A(x),B(y))$. However, we will seek the maximal fuzzy set in $\mathcal{F}(Y)$ that makes $M(x,y)$ in (6) take its maximum $R_z(A(x),B(y)) \vee R'_z(A(x),B(y))$.

Furthermore, we have the following triple I FMT theory.

THEOREM 2. TRIPLE I FMT FOR R_z. *Suppose X, Y are nonempty sets, $A \in \mathcal{F}(X)$, $B, B^* \in \mathcal{F}(Y)$. Then for Zadeh's operator R_z, the maximal fuzzy set $A^* \in \mathcal{F}(X)$ that makes $M(x,y)$ in (6) take its maximum is determined by*

$$A^*(x) = A_0^*(x)\chi_{E_x} + \chi_{E_x^c}, \tag{10}$$

where χ_M denotes the characteristic function of the set M which is defined as follows:

$$\chi_M(y) = \begin{cases} 1, & y \in M, \\ 0, & y \notin M, \end{cases}$$

and

$$A_0^*(x) = \inf_{\substack{y \in E_x \\ R_z(A(x),B(y))>1/2}} R'_z(A(x),B(y)), \qquad E_x = \{y \in Y \mid B^*(y) < R_z(A(x),B(y))\}. \tag{11}$$

PROOF. First, we show that A^* determined by (10) satisfies (6), i.e., for any $y \in Y$, $M(x,y)$ in (6) takes its maximum $R_z(A(x),B(y)) \vee R'_z(A(x),B(y))$.

We shall discuss in two possible cases as follows.

(1) If $y \in E_x$ with $R_z(A(x),B(y)) > 1/2$, then

$$A^*(x) \le R'_z(A(x),B(y)). \tag{12}$$

By the definition of Zadeh's operator R_z, we know $R_z(A^*(x),B^*(y)) \ge (A^*(x))'$, and using (12), we have

$$R_z(A(x),B(y)) \le R_z(A^*(x),B^*(y)).$$

Thus, A^* makes $M(x,y)$ in (6) take its maximum $R_z(A(x),B(y)) \vee R'_z(A(x),B(y))$.

(2) If $y \notin E_x$, then $A^*(x) \equiv 1$. Furthermore, we have

$$\begin{aligned} M(x,y) &= (A(x) \to B(y)) \to (A^*(x) \to B^*(y)) = (A(x) \to B(y)) \to (1 \to B^*(y)) \\ &= (A(x) \to B(y)) \to B^*(y). \end{aligned} \tag{13}$$

From $y \notin E_x$, we have $B^*(y) \ge R_z(A(x),B(y))$; using (13), we have

$$M(x,y) = R_z(A(x),B(y)) \vee R'_z(A(x),B(y)).$$

That is, A^* makes $M(x,y)$ in (6) take its maximum.

(3) If $R_z(A(x),B(y)) \le 1/2$, then $R'_z(A(x),B(y)) \ge 1/2$ and

$$R_z(A(x),B(y)) \vee R'_z(A(x),B(y)) = R'_z(A(x),B(y)).$$

This implies

$$M(x,y) = R'_z(A(x),B(y)). \tag{14}$$

We discuss this in two possible cases.

(a) If $R_z(A(x),B(y)) \le R_z(A^*(x),B^*(y))$, then (14) obviously holds.

(b) If $R_z(A(x),B(y)) > R_z(A^*(x),B^*(y))$, then

$$M(x,y) = R'_z(A(x),B(y)) \vee R_z(A^*(x),B^*(y)) = R'_z(A(x),B(y)).$$

Hence, $M(x,y)$ always takes its maximum.

Now, we prove that $A^* \in \mathcal{F}(X)$ will be the maximal fuzzy set that makes $M(x, y)$ in (6) take its maximum. Otherwise, there exists $D \in \mathcal{F}(X)$ with $D(x_0) > A^*(x_0)$ for some $x_0 \in X$. From A^* determined by (10), there exists $y_0 \in E_{x_0}$ such that $R_z(A(x_0), B(y_0)) > 1/2$ and

$$D(x_0) > R'_z(A(x_0), B(y_0)). \tag{15}$$

Then, we will verify that

$$R_z(A(x_0), B(y_0)) > R_z(D(x_0), B^*(y_0)). \tag{16}$$

In fact, we can discuss two possible cases.

(a) If $D(x_0) \leq B^*(y_0)$, then

$$R_z(D(x_0), B^*(y_0)) = D(x_0) \vee D'(x_0).$$

Since $y_0 \in E_{x_0}$, we know $B^*(y_0) < R_z(A(x_0), B(y_0))$, and hence $D(x_0) < R_z(A(x_0), B(y_0))$. Using (15), we have $D'(x_0) < R_z(A(x_0), B(y_0))$. This implies that (16) holds.

(b) If $D(x_0) > B^*(y_0)$, then

$$R_z(D(x_0), B^*(y_0)) = D'(x_0) \vee B^*(y_0).$$

And (15) leads to $D'(x_0) < R_z(A(x_0), B(y_0))$. By $y_0 \in E_{x_0}$, we have

$$B^*(y_0) < R_z(A(x_0), B(y_0)).$$

Thus, we proved also that (16) holds. Furthermore, by $R_z(A(x_0), B(y_0)) > 1/2$, it follows that

$$\begin{aligned} R_z(A(x_0), B(y_0)) \to R_z(D(x_0), B^*(y_0)) &= R'_z(A(x_0), B(y_0)) \vee R_z(D(x_0), B^*(y_0)) \\ &< R_z(A(x_0), B(y_0)). \end{aligned} \tag{17}$$

That is, D will make $M(x, y)$ in (6) not take its maximum. ∎

REMARK 4. It is clear that for any $D \in \mathcal{F}(X)$ with $D < A^*$, then D may make $M(x, y)$ in (6) not take its maximum $R_z(A(x), B(y)) \vee R'_z(A(x), B(y))$. This is because Zadeh's operator R_z does not have the monotonicity property with respect to the first variables. However, we have following result.

COROLLARY 3. *Under the hypothesis of Theorem 2, suppose that there exists $D \in \mathcal{F}(X)$ with $D < A^*$. Then D makes $M(x, y)$ in (6) take its maximum if and only if $R_z(A(x), B(y)) \leq 1/2$, or $R_z(A(x), B(y)) \leq D(x) \vee D'(x)$ when $D(x) \leq B^*(y)$ for $x \in X$, $y \notin E_x$.*

PROOF.

SUFFICIENCY. By the proof of Theorem 2, for any $x \in X$, $y \in Y$, if $R_z(A(x), B(y)) \leq 1/2$, then any $D \in \mathcal{F}(X)$ will make $M(x, y)$ in (6) take its maximum. If

$$R_z(A(x), B(y)) \leq D(x) \vee D'(x), \tag{18}$$

when $D(x) \leq B^*(y)$ for $x \in X$, $y \notin E_x$, then D will make $M(x, y)$ in (6) take its maximum. In fact, if $y \in E_x$, from the hypothesis of this corollary and the proof of Theorem 2, it is easy to see $R_z(A(x), B(y)) \leq R_z(D(x), B^*(y))$. That is, D will make $M(x, y)$ in (6) take its maximum.

If $y \notin E_x$, then $B^*(y) \geq R_z(A(x), B(y))$. Next, we will discuss two possible cases.

(1) If $D(x) > B^*(y)$, then

$$\begin{aligned} M(x, y) &= (A(x) \to B(y)) \to (D(x) \to B^*(y)) = R_z(A(x), B(y)) \to (D'(x) \vee B^*(y)) \\ &= R_z(A(x), B(y)) \vee R'_z(A(x), B(y)). \end{aligned} \tag{19}$$

So, D will make $M(x, y)$ in (6) take its maximum.

(2) If $D(x) \leq B^*(y)$, then from (18), we have

$$\begin{aligned} M(x, y) &= (A(x) \to B(y)) \to (D(x) \to B^*(y)) = R_z(A(x), B(y)) \to (D(x) \vee D'(x)) \\ &= R_z(A(x), B(y)) \vee R'_z(A(x), B(y)). \end{aligned} \tag{20}$$

Similarly, D will make $M(x, y)$ in (6) take its maximum.

NECESSITY. If there exists $x_0 \in X$ and $y_0 \notin E_{x_0}$ such that $R_z(A(x_0), B(y_0)) > 1/2$ and $R_z(A(x_0), B(y_0)) > D(x_0) \vee D'(x_0)$ with $D(x_0) \leq B^*(y_0)$, then we have

$$\begin{aligned}(A(x_0) \to B(y_0)) \to (D(x_0) \to B^*(y_0)) &= R_z(A(x_0), B(y_0)) \to (D(x_0) \vee D'(x_0)) \\ &= R'_z(A(x_0), B(y_0)) \vee D(x_0) \vee D'(x_0) < R_z(A(x_0), B(y_0)).\end{aligned} \tag{21}$$

Hence, D will make $M(x, y)$ in (6) not take its maximum. This contradicts the hypothesis of this corollary. ∎

3. THE REVERSIBILITY PROPERTIES OF THE TRIPLE I METHOD FOR R_z

This section is devoted to discussing the reversibility properties of the triple I method for Zadeh's implication operator R_z. To do this, we give the following several theorems.

THEOREM 4. *Suppose that X, Y are nonempty sets, $A, A^* \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$, and A is a normal fuzzy set (i.e., there exists $x_0 \in X$ such that $A(x_0) = 1$). Then for the triple I FMP method with R_z, $A^* = A$ implies*

$$B^*(y) = \begin{cases} 0, & \text{if } B(y) \leq \dfrac{1}{2}, \\ B(y), & \text{if } B(y) > \dfrac{1}{2}, \end{cases} \qquad y \in Y.$$

PROOF. Suppose $A^* = A$. Then from (9) we have

$$E_y = \{x \in X \mid A'(x) < R_z(A(x), B(y))\}. \tag{22}$$

If $B(y) > 1/2$, using the normal property of A, there exists $x_0 \in X$ such that $A(x_0) = 1$. So, from $A'(x_0) = 0 < 1/2 < B(y) = R_z(A(x_0), B(y))$ with B^* determined by (8), we have $B^*(y) \geq A(x_0) \wedge R_z(A(x_0), B(y)) = B(y)$.

Next, we verify

$$B^*(y) \leq B(y). \tag{23}$$

For any $x \in E_y$ with $R_z(A(x), B(y)) > 1/2$, if $A(x) \leq B(y)$, then $R_z(A(x), B(y)) = A'(x) \vee A(x)$. From $x \in E_y$, we know $A'(x) < A(x)$, and hence $R_z(A(x), B(y)) = A(x)$. From B^* determined by (8) again, we have $B^*(y) \leq B(y)$.

If $A(x) > B(y)$, then $R_z(A(x), B(y)) = A'(x) \vee B(y)$. From $x \in E_y$, we have $A'(x) < B(y)$. So, $R_z(A(x), B(y)) = B(y)$, and using B^* determined by (8), we get $B^*(y) \leq B(y)$. The above proof shows that $B^*(y) = B(y)$ when $B(y) > 1/2$.

On the other hand, by the proof of (23), if there exists $x \in E_y$ such that $R_z(A(x), B(y)) > 1/2$, then we have $B(y) > 1/2$. Consequently,

$$E_y \cap \left\{x \in X \mid R_z(A(x), B(y)) > \frac{1}{2}\right\} = \Phi, \qquad \text{when } B(y) \leq \frac{1}{2}.$$

Using $B^*(y)$ determined by (8) again, we have $B^*(y) = 0$, when $B(y) \leq 1/2$. The proof is completed. ∎

From Theorem 4, we can obtain immediately the following criterion of reversibility for the triple I FMP with R_z.

COROLLARY 5. *Suppose X, Y are nonempty sets, $A, A^* \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$, and A is a normal fuzzy set (i.e., there exists $x_0 \in X$ such that $A(x_0) = 1$). Then for the triple I FMP method with R_z, $A^* = A$ implies $B^* = B$ if and only if $B(y) > 1/2$ or $B(y) = 0$.*

THEOREM 6. *Suppose X, Y are nonempty sets, $A \in \mathcal{F}(X)$, $B, B^* \in \mathcal{F}(Y)$, and B' is a normal fuzzy set. Then by the triple I FMT method with R_z, $B^* = B$ implies*

$$A^*(x) = \begin{cases} A(x), & \text{if } A(x) < \dfrac{1}{2}, \\ 1, & \text{if } A(x) \geq \dfrac{1}{2}, \end{cases} \qquad x \in X.$$

PROOF. Suppose that $B^* = B$. Then by (11), we get

$$E_x = \{y \in Y \mid B(y) < R_z(A(x), B(y))\}. \tag{24}$$

If $A(x) < 1/2$, by the normal property of B', there exists $y_0 \in Y$ such that $B'(y_0) = 1$. So, by $B(y_0) = 0 < 1/2 < A'(x) = R_z(A(x), B(y_0))$ and A^* determined by (10), we have

$$A^*(x) \leq R_z'(A(x), B(y_0)) = A(x).$$

In what follows, we verify

$$A^*(x) \geq A(x). \tag{25}$$

For any $y \in E_x$ satisfying $R_z(A(x), B(y)) > 1/2$, if $A(x) \leq B(y)$, then $R_z(A(x), B(y)) = A'(x) \vee A(x)$. By $y \in E_x$, we know $B(y) < A'(x)$. Hence, $R_z(A(x), B(y)) = A'(x)$, and using $A^*(x)$ determined by (10), we have $A^*(x) \geq A(x)$.

If $A(x) > B(y)$, then $R_z(A(x), B(y)) = A'(x) \vee B(y)$. From $y \in E_x$, we know $B(y) < A'(x)$, so that $R_z(A(x), B(y)) = A'(x)$. Using A^* determined by (10), we know $A^*(x) \geq A(x)$. All of these show that $A^*(x) = A(x)$ when $A(x) < 1/2$.

On the other hand, by the proof of (25), it can be seen that if there exists $y \in E_x$ such that $R_z(A(x), B(y)) > 1/2$, then we have $A(x) < 1/2$, so that $E_x \cap \{x \in X \mid R_z(A(x), B(y)) > 1/2\} = \Phi$, when $A(x) \geq 1/2$. Using $A^*(x)$ determined by (10), we get $A^*(x) = 1$, when $A(x) \geq 1/2$. The proof is completed. ∎

From Theorem 6, it is easy to give the criterion of reversibility of the triple I FMT method for R_z.

COROLLARY 7. *Suppose X, Y are nonempty sets, $A \in \mathcal{F}(X)$, $B, B^* \in \mathcal{F}(Y)$, and B' is a normal fuzzy set. Then for the triple I FMT method, $B^* = B$ implies $A^* = A$ if and only if $A(x) < 1/2$ or $A(x) = 1$.*

4. THE FORMULAE OF THE α-TRIPLE I FMP AND FMT FOR R_z

Now, let us consider the generalized problem of the triple I method; i.e., for $\alpha \in [0, 1]$, our purpose is to seek the optimal solution satisfying (7). To do this, we will give the following α-triple I principles.

Principle of the α-Triple I FMP

Suppose that X and Y are nonempty sets, $A, A^* \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$. If $B^* \in \mathcal{F}(Y)$ satisfying (2) is the minimal fuzzy set provided (7), then it is called an α-solution of (2) for the triple I FMP.

Principle of the α-Triple I FMT

Suppose that X and Y are nonempty sets, $A \in \mathcal{F}(X)$, $B, B^* \in \mathcal{F}(Y)$. If $A^* \in \mathcal{F}(X)$ satisfying (5) is the maximal fuzzy set provided (7), then it is called an α-solution of (5) for the triple I FMT.

In the following, we will give several theorems for the α-triple I FMP and α-triple I FMT, respectively.

THEOREM 8. *Suppose that X and Y are nonempty sets, $A, A^* \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$. Then, there exists $B^* \in \mathcal{F}(Y)$ satisfying (7) if and only if*

$$N(x,y) = R_z(A(x), B(y)) \to (A^*(x) \vee (A^*(x))') \geq \alpha. \tag{26}$$

PROOF.

SUFFICIENCY. Taking $B^*(y) \equiv 1$, then by (26), we know B^* satisfies (7).

NECESSITY. If B^* satisfies (7), then noting that Zadeh's operator R_z is increasing with respect to the second variable, we have

$$\begin{aligned} N(x,y) &= R_z(A(x), B(y)) \to (A^*(x) \vee (A^*(x))') = R_z(A(x), B(y)) \to R_z(A^*(x), 1) \\ &\geq R_z(A(x), B(y)) \to R_z(A^*(x), B^*(y)) \geq \alpha. \end{aligned}$$

That is, (26) holds. ∎

THEOREM 9. α-TRIPLE I FMP FOR R_z. *Suppose X and Y are nonempty sets, $A, A^* \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$, and (26) holds. Then, for the triple I FMP method with Zadeh's operator R_z, the minimal fuzzy set $B^* \in \mathcal{F}(Y)$ satisfying (7) is determined by*

$$B^*(y) = \sup_{x \in E_y \cap K_y} [A^*(x) \wedge R_z(A(x), B(y))] \wedge \alpha, \qquad y \in Y, \tag{27}$$

where $E_y = \{x \in X \mid (A^*(x))' < R_z(A(x), B(y))\}$, and

$$K_y = \{x \in X \mid A^*(x) \wedge R_z(A(x), B(y)) > \alpha'\}. \tag{28}$$

PROOF.

(I) For any $y \in Y$ with $x \in E_y \cap K_y$, then we have

$$B^*(y) \geq A^*(x) \wedge R_z(A(x), B(y)) \wedge \alpha. \tag{29}$$

Since R_z has the property of preserving the intersection operator with respect to the second variable, we get

$$\begin{aligned} M(x,y) &= R_z(A(x), B(y)) \to R_z(A^*(x), B^*(y)) \\ &\geq R_z(A(x), B(y)) \to R_z(A^*(x), A^*(x) \wedge R_z(A(x), B(y)) \wedge \alpha) \\ &= R_z(A(x), B(y)) \to R_z(A^*(x), A^*(x)) \wedge R_z(A^*(x), R_z(A(x), B(y))) \wedge R_z(A^*(x), \alpha) \\ &= P(x,y) \wedge Q(x,y) \wedge S(x,y), \end{aligned}$$

where

$$\begin{aligned} P(x,y) &= R_z(A(x), B(y)) \to R_z(A^*(x), A^*(x)), \\ S(x,y) &= R_z(A(x), B(y)) \to R_z(A^*(x), \alpha), \\ Q(x,y) &= R_z(A(x), B(y)) \to R_z(A^*(x), R_z(A(x), B(y))). \end{aligned}$$

Now, we will prove $M(x,y) \geq \alpha$.

First, using the hypothesis in (26), we have $P(x,y) \geq \alpha$. Now, we verify $Q(x,y) \geq \alpha$ in two different cases.

(1) If $A^*(x) \leq R_z(A(x), B(y))$, then

$$R_z(A^*(x), R_z(A(x), B(y))) = A^*(x) \vee (A^*(x))'.$$

Applying the hypothesis in (26), we have $Q(x,y) \geq \alpha$.

(2) If $A^*(x) > R_z(A(x), B(y))$, then

$$R_z(A^*(x), R_z(A(x), B(y))) = (A^*(x))' \vee R_z(A(x), B(y)).$$

By the definition of R_z, we know $Q(x,y) = R_z(A(x),B(y)) \vee R_z'(A(x),B(y))$. By the hypothesis in (26) again, we have

$$Q(x,y) = R_z(A(x),B(y)) \vee R_z'(A(x),B(y)) = N(x,y) \geq \alpha.$$

In the following, we show $S(x,y) \geq \alpha$ in two possible cases.

(1) If $A^*(x) \leq \alpha$, then $R_z(A^*(x),\alpha) = A^*(x) \vee (A^*(x))'$. Using the hypothesis in (26), we have $S(x,y) \geq \alpha$.

(2) If $A^*(x) > \alpha$, then $R_z(A^*(x),\alpha) = (A^*(x))' \vee \alpha$. Consequently,

$$S(x,y) = R_z(A(x),B(y)) \to (A^*(x))' \vee \alpha. \tag{30}$$

We discuss two possible cases.

(a) If $R_z(A(x),B(y)) \leq (A^*(x))' \vee \alpha$, then from (30), we know

$$S(x,y) = R_z(A(x),B(y)) \vee R_z'(A(x),B(y)). \tag{31}$$

Moreover, noting that $R_z(A(x),B(y)) \leq (A^*(x))' \vee A^*(x)$ with the hypothesis in (26), we get

$$S(x,y) = R_z(A(x),B(y)) \vee R_z'(A(x),B(y)) = N(x,y) \geq \alpha.$$

(b) If $R_z(A(x),B(y)) > (A^*(x))' \vee \alpha$, then from (30), we have

$$S(x,y) = R_z'(A(x),B(y)) \vee (A^*(x))' \vee \alpha \geq \alpha. \tag{32}$$

(II) For any $y \in Y$ with $x \notin E_y$, then we have

$$R_z(A(x),B(y)) \leq (A^*(x))'. \tag{33}$$

Next, we show $M(x,y) \geq \alpha$. In fact:

(1) If $A^*(x) \leq B^*(y)$, then by $R_z(A^*(x),B^*(y)) = (A^*(x))' \vee A^*(x)$ with the hypothesis in (26), we have $M(x,y) \geq \alpha$.

(2) If $A^*(x) > B^*(y)$, then from $R_z(A^*(x),B^*(y)) = (A^*(x))' \vee B^*(x)$ with (33), we know $M(x,y) = R_z(A(x),B(y)) \vee R_z'(A(x),B(y))$. Using the hypothesis in (26) again, we have

$$M(x,y) = R_z(A(x),B(y)) \vee R_z'(A(x),B(y)) = N(x,y) \geq \alpha.$$

(III) For any $y \in Y$ with $x \notin K_y$, then we have

$$(A^*(x))' \geq \alpha, \tag{34}$$

or

$$R_z'(A(x),B(y)) \geq \alpha. \tag{35}$$

If (34) holds, we verify $M(x,y) \geq \alpha$ in two possible cases.

(1) If $A^*(x) \leq B^*(y)$, then similar to (1) in (II) it follows that $M(x,y) \geq \alpha$.

(2) If $A^*(x) > B^*(y)$, then $R_z(A^*(x),B^*(y)) = (A^*(x))' \vee B^*(x)$. Hence,

$$M(x,y) = R_z(A(x),B(y)) \to (A^*(x))' \vee (B^*(y)). \tag{36}$$

We discuss two possible cases.

(a) If $R_z(A(x), B(y)) \le (A^*(x))' \vee (B^*(y))$, then

$$M(x,y) = R_z(A(x), B(y)) \vee R'_z(A(x), B(y)).$$

Furthermore, using the hypothesis in (26), we have

$$M(x,y) = R_z(A(x), B(y)) \vee R'_z(A(x), B(y)) = N(x,y) \ge \alpha.$$

(b) If $R_z(A(x), B(y)) > (A^*(x))' \vee (B^*(y))$, then

$$M(x,y) = R'_z(A(x), B(y)) \vee (A^*(x))' \vee (B^*(y)).$$

From (34), we know $M(x,y) \ge \alpha$.

If (35) holds, then by the definition of Zadeh's operator R_z, we have

$$M(x,y) \ge R'_z(A(x), B(y)).$$

Hence, $M(x,y) \ge \alpha$.

In summary, we proved that $B^* \in \mathcal{F}(Y)$ determined by (27) satisfies (7). Finally, we will prove that B^* is the minimal fuzzy set satisfying (7). Otherwise, there exists $D \in \mathcal{F}(Y)$ and $y_0 \in Y$ provided $D(y_0) < B^*(y_0)$. Using B^* determined by (27), there exists $x_0 \in E_{y_0} \cap K_{y_0}$ such that

$$D(y_0) < A^*(x_0) \wedge R_z(A(x_0), B(y_0)) \wedge \alpha. \tag{37}$$

Therefore, from $D(y_0) < A^*(x_0)$, we have $R_z(A^*(x_0), D(y_0)) = (A^*(x_0))' \vee D(y_0)$. From (37) we have also $D(y_0) < R_z(A(x_0), B(y_0))$. As $x_0 \in E_{y_0}$, we get

$$R_z(A(x_0), B(y_0)) > (A^*(x_0))' \vee D(y_0).$$

Furthermore, (37) leads to $D(y_0) < \alpha$. And, from $x_0 \in K_{y_0}$, we have

$$\begin{aligned} R_z(A(x_0), B(y_0)) \to R_z(A^*(x_0), D(y_0)) &= R'_z(A(x_0), B(y_0)) \vee (A^*(x_0))' \vee D(y_0) \\ &= (R_z(A(x_0), B(y_0)) \wedge A^*(x_0))' \vee D(y_0) < \alpha. \end{aligned} \tag{38}$$

Equation (38) shows that D will not satisfy (7). The proof is completed. ∎

REMARK 5. As Zadeh's operator R_z is increasing with respect to the second variable, so, under the hypotheses of Theorem 9, for any $B_0^*(y) \in \mathcal{F}(Y)$ satisfying $B^*(y) \le B_0^*(y) \le 1$, then $B_0^*(y)$ satisfies (7), where $B^*(y)$ is determined by (27).

THEOREM 10. *Suppose that X and Y are nonempty sets, $A \in \mathcal{F}(X)$, $B, B^* \in \mathcal{F}(Y)$. Then, there exists $A^* \in \mathcal{F}(X)$ satisfying (7) if and only if*

$$R_z(A(x), B(y)) \vee R'_z(A(x), B(y)) \ge \alpha. \tag{39}$$

PROOF.

SUFFICIENCY. Taking $A^*(x) \equiv 0$, then by (39), we know that A^* satisfies (7).

NECESSITY. Suppose that A^* satisfies (7). Since R_z is an increasing operator with respect to the second variable, we have

$$\begin{aligned} R_z(A(x), B(y)) \vee R'_z(A(x), B(y)) &= R_z(A(x), B(y)) \to 1 \\ &\ge R_z(A(x), B(y)) \to R_z(A^*(x), B^*(y)) \ge \alpha. \end{aligned}$$

That is, (39) holds. ∎

THEOREM 11. α-TRIPLE I FMT FOR R_z. *Suppose that X and Y are nonempty sets, $A \in \mathcal{F}(X)$, $B, B^* \in \mathcal{F}(Y)$, and (39) holds. Then, for the triple I FMT method with Zadeh's operator R_z, the maximal fuzzy set $A^* \in \mathcal{F}(X)$ satisfying (7) is determined by*

$$A^*(x) = A_1^*(x)\chi_{K_x} + \chi_{K_x^c}, \tag{40}$$

$$A_1^*(x) = \inf_{\substack{y \in K_x \\ R_z(A(x),B(y))>1/2}} R'_z(A(x), B(y)) \vee \alpha', \qquad \text{and}$$

$$K_x = \{y \in Y \mid B^*(y) \vee R'_z(A(x), B(y)) < \alpha\}. \tag{41}$$

PROOF. First, we prove that $A^*(x)$ determined by (40) satisfies (7). For any $y \in Y$, we will discuss two possible cases.

(1) For $y \in K_x$ with $R_z(A(x), B(y)) > 1/2$, then we have

$$A^*(x) \leq R'_z(A(x), B(y)) \vee \alpha'. \tag{42}$$

If $A^*(x) \leq R'_z(A(x), B(y))$, then from $R_z(A^*(x), B^*(y)) \geq (A^*(x))'$, we know

$$R_z(A^*(x), B^*(y)) \geq R_z(A(x), B(y)).$$

This implies

$$M(x,y) = R_z(A(x), B(y)) \to R_z(A^*(x), B^*(y)) = R_z(A(x), B(y)) \vee R'_z(A(x), B(y)).$$

By the hypothesis of the theorem, we get $M(x,y) \geq \alpha$.

If $A^*(x) \leq \alpha'$, then we discuss two possible cases.

(a) If $R_z(A(x), B(y)) \leq R_z(A^*(x), B^*(y))$, then using the hypothesis of the theorem and the definition of Zadeh's operator R_z, we have $M(x,y) \geq \alpha$.

(b) If $R_z(A(x), B(y)) > R_z(A^*(x), B^*(y))$, then we have

$$M(x,y) = R'_z(A(x), B(y)) \vee R_z(A^*(x), B^*(y)).$$

From $R_z(A^*(x), B^*(y)) \geq (A^*(x))' \geq \alpha$, it follows that $M(x,y) \geq \alpha$.

(2) For $y \notin K_x$, then $A^*(x) \equiv 1$. So, we have

$$M(x,y) = R_z(A(x), B(y)) \to R_z(A^*(x), B^*(y)) = R_z(A(x), B(y)) \to B^*(y). \tag{43}$$

If $R_z(A(x), B(y)) \leq B^*(y)$, then by the hypothesis of this theorem, we have

$$M(x,y) = R_z(A(x), B(y)) \vee R'_z(A(x), B(y)) \geq \alpha.$$

If $R_z(A(x), B(y)) > B^*(y)$, then by $y \notin K_x$, we obtain

$$M(x,y) = R'_z(A(x), B(y)) \vee B^*(y) \geq \alpha.$$

(3) If $R_z(A(x), B(y)) \leq 1/2$, then $R'_z(A(x), B(y)) \geq 1/2$. Similar to the proof of the corresponding part of Theorem 2, we know $M(x,y) = R'_z(A(x), B(y))$. Using the hypothesis in (39), we have $M(x,y) \geq \alpha$.

Next, we will prove that $A^* \in \mathcal{F}(X)$ is the maximal fuzzy set satisfying (7). Otherwise, there exists $D \in \mathcal{F}(X)$ with $D(x_0) > A^*(x_0)$ where $x_0 \in X$. By A^* determined by (40), there exists $y_0 \in K_{x_0}$ such that $R_z(A(x_0), B(y_0)) > 1/2$ with

$$D(x_0) > R'_z(A(x_0), B(y_0)) \vee \alpha'. \tag{44}$$

This implies

$$R_z(A(x_0), B(y_0)) \to R_z(D(x_0), B^*(y_0)) < \alpha. \tag{45}$$

That is, D does not satisfy (7). The proof of (45) can be discussed in two possible cases.

(a) If $D(x_0) \le B^*(y_0)$, then $R_z(D(x_0), B^*(y_0)) = D(x_0) \vee D'(x_0)$. From $y_0 \in K_{x_0}$, we know $B^*(y_0) < \alpha$. By the hypothesis in (39), we know $R_z(A(x_0), B(y_0)) \ge \alpha$. Hence,

$$D(x_0) < R_z(A(x_0), B(y_0)). \tag{46}$$

By using (44), we have

$$D'(x_0) < R_z(A(x_0), B(y_0)). \tag{47}$$

Combining (46) and (47), it follows that $R_z(D(x_0), B^*(y_0)) < R_z(A(x_0), B(y_0))$. Furthermore, we have

$$\begin{aligned} R_z(A(x_0), B(y_0)) \to R_z(D(x_0), B^*(y_0)) &= R_z'(A(x_0), B(y_0)) \vee D(x_0) \vee D'(x_0) \\ &\le R_z'(A(x_0), B(y_0)) \vee B^*(y_0) \vee D'(x_0). \end{aligned}$$

By $y_0 \in K_{x_0}$ with (44), it follows that (45) holds.

(b) If $D(x_0) > B^*(y_0)$, then $R_z(D(x_0), B^*(y_0)) = D'(x_0) \vee B^*(y_0)$. Similar to the discussion in the above case (a), we have $B^*(y_0) < R_z(A(x_0), B(y_0))$. And from (44), we have $D'(x_0) < R_z(A(x_0), B(y_0))$. Hence,

$$R_z(A(x_0), B(y_0)) > R_z(D(x_0), B^*(y_0)). \tag{48}$$

In addition, we obtain

$$R_z(A(x_0), B(y_0)) \to R_z(D(x_0), B^*(y_0)) = R_z'(A(x_0), B(y_0)) \vee D'(x_0) \vee B^*(y_0). \tag{49}$$

Furthermore, from $y_0 \in K_{x_0}$ with (44), it follows that (45) holds.

All of these show that $A^* \in \mathcal{F}(X)$ is the maximal fuzzy set satisfying (7). The proof is completed. ∎

REMARK 6. As Zadeh's operator R_z does not have the monotonicity with respect to its first variable, so, for any $D \in \mathcal{F}(X)$ with $D < A^*$ for $y \in Y$, D may not satisfy (7). Thus, we have the following result.

COROLLARY 12. *Under the hypotheses of Theorem 11, suppose that there exists $D \in \mathcal{F}(X)$ with $D < A^*$. Then D satisfies (7) if and only if $R_z(A(x), B(y)) \le 1/2$ or $R_z(A(x), B(y)) \le D(x) \vee D'(x)$ or $D(x) \vee D'(x) \ge \alpha$, when $D(x) \le B^*(y)$ whenever $x \in X$, $y \notin K_x$.*

PROOF. SUFFICIENCY. By the proof of Theorem 11, for any $x \in X$ and $y \in Y$, if $R_z(A(x), B(y)) \le 1/2$, then for any $D \in \mathcal{F}(X)$, we have

$$R_z(A(x), B(y)) \to R_z(D(x), B^*(y)) = R_z'(A(x), B(y)).$$

By the hypothesis in (39), we know that D satisfies (7). If

$$R_z(A(x), B(y)) \le D(x) \vee D'(x), \qquad \text{or} \qquad D(x) \vee D'(x) \ge \alpha, \tag{50}$$

when $D(x) \le B^*(y)$ for any $x \in X$, $y \notin K_x$, then, we will prove that D satisfies (7).

In fact, if $y \in K_x$, then by the hypothesis of Corollary 12 and the proof of Theorem 11, we have

$$R_z(A(x), B(y)) \to R_z(D(x), B^*(y)) \ge \alpha.$$

That is, D satisfies (7). If $y \notin K_x$ with $D(x) > B^*(y)$, then

$$(A(x) \to B(y)) \to (D(x) \to B^*(y)) = R_z(A(x), B(y)) \to (D'(x) \vee B^*(y)). \tag{51}$$

Similar to the proof of the corresponding part of Theorem 11, D also satisfies (7).

If $y \notin K_x$ with $D(x) \le B^*(y)$, then by the hypothesis of Corollary 12, we have

$$(A(x) \to B(y)) \to (D(x) \to B^*(y)) = R_z(A(x), B(y)) \to (D(x) \vee D'(x)). \tag{52}$$

If $R_z(A(x), B(y)) \le D(x) \vee D'(x)$, then due to the hypothesis in (39), D satisfies (7). If $R_z(A(x), B(y)) > D(x) \vee D'(x) \ge \alpha$, then, similarly, it can deduced that D satisfies (7).

NECESSITY. If there exists $x_0 \in X$ and $y_0 \notin K_{x_0}$ such that $R_z(A(x_0), B(y_0)) > 1/2$ and $R_z(A(x_0), B(y_0)) \wedge \alpha > D(x_0) \vee D'(x_0)$ when $D(x_0) \le B^*(y_0)$, then we have

$$\begin{aligned}(A(x_0) \to B(y_0)) \to (D(x_0) \to B^*(y_0)) &= R_z(A(x_0), B(y_0)) \to (D(x_0) \vee D'(x_0)) \\ &= R'_z(A(x_0), B(y_0)) \vee D(x_0) \vee D'(x_0) \\ &= D(x_0) \vee D'(x_0) < \alpha.\end{aligned}$$

So, D does not satisfy (7). This contradicts the hypothesis of Corollary 12. ∎

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